# Probability Distributions

The Distribution | Binomial | Geometric | Hypergeometric | Using Excel | Advanced applications

What is the probability of rolling three 5's in a row with a six-sided die?

What if you didn't know what order they could occur in?

? What is the probability of rolling three 5's with seven tosses of a six-sided die?

# Probability Distributions

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Who might want to know the answer to these questions, and why?

How likely is it to have more than one defective part in this product package?

How much do we have to charge for this lottery ticket to make a profit?

How many doses are most likely to be sufficient to cure this patient? How much can we expect it to cost?

What's the likelihood of more 911 calls coming in then we have ambulances for?


# Where are we?


# Conceptualizing a "Distribution"

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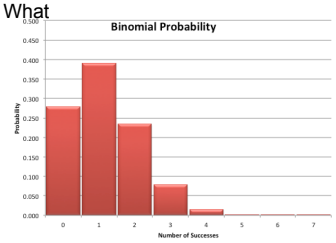
What is the probability of rolling three 5's in a row with a six-sided die?

? What is the probability of rolling three 5's with seven tosses of a six-sided die?

What's different between these two questions; what's the same?

Is three fives in seven tosses the only thing that is possible? What else could happen?

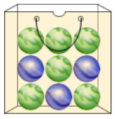



This is a **discrete probability** distribution. Any **whole** number of successes is possible. However, the likelihood varies with the probability of success (P).

If the probability of success were to rise, what do you think would happen to the appearance of this graph? Why?

## Achtung!

What is the probability of drawing out 2 blue then 1 green marble from this bag (with replacement)?



**Probability Calculation**
**(Using techniques from unit two)**

What is the probability of drawing out 2 blue and 1 green marble from this bag in any order (with replacement)?

## Conceptualizing a "Distribution"



Can you extend the solution to this problem on the right?

? What is the probability of rolling three 5's with seven tosses of a six-sided die?

## Conceptualizing a "Distribution"



Can you come up with a formula for this kind of problem, and its conditions for use?

? What is the probability of rolling three 5's with seven tosses of a six-sided die?

*Hint for conditions*: what was significant about the way in which we drew out the marbles and how is that like tossing a die?

$$P(x) = {}_nC_r \times P^r \times Q^{n-r}$$

## Solution

What is the probability of rolling three 5's in a row with a six-sided die?

? What is the probability of rolling three 5's with seven tosses of a six-sided die?

$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$

$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$ ... woah, now what?

$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

But, did it have to happen in this order?

There's another way!

$\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

How many more are there?

Does this seem familiar to you?

**? What is the probability of rolling three 5's with seven tosses of a six-sided die?**

Isn't this just permutations of identical items (i.e. combinations)?

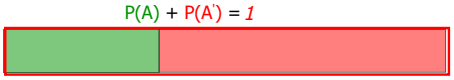
How?

These are probabilities of success and failure on each individual trial...

| Probability of Success | Probability of Failure |
|---|---|
| $\frac{1}{6}$ | $\frac{5}{6}$ |

Can you have anything besides a 5 or not a 5?

$P(A) + P(A') = 1$



$P(A)$ $P(A') = 1 - P(A)$

**? What is the probability of rolling three 5's with seven tosses of a six-sided die?**

$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

P P P Q Q Q Q

To arrange these identical items, we're just permuting the letters p and Q (success and failure).

To arrange these identical items, we're just choosing P trials to be successes, and Q trials to be failures.

$$\frac{7!}{3![\times 4!]} = \frac{n!}{r!(n-r)!} = {}_nC_r = {}_7C_3$$

$= 35$

That means there are 35 arrangements of 3 successes and 4 failures (in seven trials).

**? What is the probability of rolling three 5's with seven tosses of a six-sided die?**

But wait, we still don't know the probability of 3 successes happening!

We know this: And we know this:

$35 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = 7.8\%$

arrangements P P P Q Q Q Q

Or, in general...

this is the #Q

$$P(x) = {}_nC_r \times p^r \times Q^{n-r}$$

## Binomial Probability: Warm up

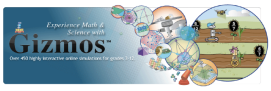
1. The faces of a 12-sided die are numbered from 1 to 12. What is the probability of rolling 9 at least twice in ten tries?

2. A coin is tossed ten times. Find the probability that
   a) exactly four heads are tossed
   b) at least two heads are tossed
   c) no more than two tails are tossed

3. In a multiple choice test that contains ten questions with each question having five possible answers, what is the probability that
   a) Colin will pass the test if he merely guesses at each question?
   b) Diane will get an "A" on the the test if she has studied and she feels that her probability of answering each question correctly is 0.75?

4. Assuming that the chance of giving birth to a girl or boy is even, what are the chances that
   a) a couple planning to have three children will have all girls?
   b) a couple planning to have five children will have at least one girl?

## What is a distribution




# Expected Value

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## Expected Outcomes: Successes vs. Values

### Successes

How many times should you should have success in a certain number of trials?

Just multiply the chance of success each trial, times the number of trials.

$$E(X) = nP$$

Note: this formula is not always used for all distributions (they each have their own).

A simple example:

5. What if you toss a coin 10 times, how many 'heads' would you expect to get?

## Expected Outcomes: Successes vs. Values

### Values i) a single outcome

What is the value outcome most likely to occur in a trial?

$$E(x) = x \bullet P(x)$$

Note: this formula is always used for all distributions (the last one is not). You just find P(x) is calculated using the appropriate formula for the distribution.

Just multiply the chance of success each trial, times the value resulting from a success (this is the success's proportion of the value per trial).

A simple example:

6a) If a jackpot is $1 million, and your probability of winning is 0.001%, what is your expected winnings from purchasing a ticket?

6b) What would the lottery have to charge to break even with odds like this?

## Expected Outcomes: Successes vs. Values

### Values ii) multiple outcome

What is the value outcome most likely to occur in a trial when more than one value outcome is possible? Just use the formula multiple times and add all the results together.

$$E(x) = \sum_{i=1}^{n} X_i \bullet P(x_i)$$

Note, this formula is always used for all distributions (expected successes is not). P(x) is calculated using the appropriate formula.

Just multiply the chance of success each trial, times the value resulting from a success (this is the success's proportion of the value per trial).

A simple example:

7a) If you earn $5 each time a coin toss shows a head, and you must pay $2 each time it's a tail, what are your expected winnings/losses in ten tosses?

7b) What is the most you would pay to play this game?

## Binomial Distribution (Exercises)

8. Assume that every time Quinlan, a hockey player, gets a breakaway on the opposition's net, he has a probability of 0.6 of scoring. If he averages two breakaways a game, what is the expected number of goals that he will score on breakaways in a season with 75 games?

9. In a manufacturing process, it is estimated that only 2 percent of the bolts that are machined are declared defective, that is, they are either too large or too small. In a package of 50 bolts, what is the probability that there is at least one defective bolt? How many would you expect?

10. If the probability is 0.15 that Luciana will hit a bull's eye on a dart board, what is the probability that she will get at least one bull's eye in ten attempts? How many would you expect her to get?

11. It seems that every carton of eggs at the supermarket contains at least one broken egg. If, in fact, it has been determined that 3 percent of the eggs supplied to a supermarket are cracked, what is the probability that if you buy two dozen eggs none of your eggs will be cracked? How many would you expect to be cracked?

12. Find the probability that at least three students in a class of 30 students were born on a Saturday. How many would you expect to be born on a Saturday in the class?

13. At the height of The Beatles' popularity, it was estimated that their music was played on every popular music radio station 40 percent of the time. What is the probability that if you tuned through ten such stations at any given moment at least one of the stations would be playing a Beatles song?

14. What is the relationship of the probability of a specific event, and the event's "proximity" to the expected value?

## Expected Value

More complex examples:

15. What is the expected sum of ten rolls of a single, six-sided die?

16. Your brother bets you that you can't roll more than 8 sixes in twenty rolls of a die. If you do, he'll pay you $5 per six. If you don't he wants to know what you'll pay him. What is the most you should offer?

17. What is the expected sum of ten rolls of two six-sided dice?

18. You and the dealer are playing blackjack with a brand new, well-shuffled deck. You have an ace, and the dealer has a seven. If blackjack pays $100, what is the most you should be willing to pay to receive the next card?

## Answer Clues

1) 20%
2a) 20.5%
b) 98.9%
c) 5.5%
3a) 3.3%
b) 52.6%
4a) 12.5%
b) 96.9%
5) five
6a) $10
b) $10 per ticket
7a) $15 won
b) $1.50 a toss
8) 90
9) 63.6%/you'd expect one
10) 80.3%/you'd expect one
11) 48.1%/ you'd expect less than one cracked egg
12) 82.3%/~4.2 students
13) 99.4%
14) closer = greater
15) 35
16) no more than 4¢ per six under 9 sixes(expect to receive about 13¢ total)
17) 70
18) $32

# Paul the Octopus

What are the chances?

**Euro 2008**

| Opponent | Stage | Date | Prediction | Result | Outcome |
|---|---|---|---|---|---|
| Poland | group stage | 8 June | Germany | 0-2 | Correct |
| Croatia | group stage | 12 June | Germany | 2-1 | Incorrect |
| Austria | group stage | 16 June | Germany | 0-1 | Correct |
| Portugal | quarter-finals | 19 June | Germany | 2-3 | Correct |
| Turkey | semi-finals | 25 June | Germany | 2-3 | Correct |
| Spain | final | 29 June | Germany[6] | 1-0 | Incorrect |

**World Cup 2010**

| Opponent | Stage | Date | Prediction | Result | Outcome |
|---|---|---|---|---|---|
| Australia | group stage | 13 June | Germany[39] | 4–0 | Correct |
| Serbia | group stage | 18 June | Serbia[39] | 0–1 | Correct |
| Ghana | group stage | 23 June | Germany[39] | 1–0 | Correct |
| England | round of 16 | 27 June | Germany[17] | 4–1 | Correct |
| Argentina | quarter-finals | 3 July | Germany[32] | 4–0 | Correct |
| Spain | semi-finals | 7 July | Spain[40] | 0–1 | Correct |
| Uruguay | 3rd place play-off | 10 July | Germany[41] | 3–2 | Correct |

**Results not involving Germany**

| Teams | Tournament | Stage | Date | Prediction | Result | Outcome |
|---|---|---|---|---|---|---|
| Netherlands vs. Spain | World Cup 2010 | final | 11 July | Spain[42] | 0–1 | Correct |